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The cyclic representations of the quantum algebra $U_q(osp(2,1))$ in terms of the Z^n-algebra*

Mo-Lin Ge†, Xu-Feng Liu† and Chang-Pu Sun†‡

† Theoretical Physics Division, Nankai Institute of Mathematics, Tianjin 300071, People's Republic of China§

‡ Physics Department, Northeast Normal University, Changchun 130024, People's Republic of China

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Abstract. The quantum superalgebra $U_q(osp(2,1))$ is embedded into the Z^n-algebra and its cyclic representations are presented in the non-generic case.

Quantum groups, quantum algebras and the theory of their representations are deeply rooted in the nonlinear integrable physical systems associated with the Yang–Baxter equation (YBE) [1–5]. Recently, representations of quantum universal enveloping algebras (quantum algebras) at roots of unity, especially the so-called cyclic representations, have drawn much attention [6–14], and new solutions to the YBE have been constructed through them [15, 16].

Besides quantum algebras, quantum superalgebras, as the q-deformation of the usual universal enveloping superalgebras, have also been introduced to construct solutions to the YBE in the generic case (q not a root of unity). To find new solutions associated with a quantum superalgebra, we naturally begin by looking for its new representations.

To this end, let us first recall the realization theory of quantum algebras. As we know its essence is to embed a quantum algebra, which has a 'complex' structure, into a 'simpler' algebra. One remarkable example of this method is the q-deformed boson realization of a quantum algebra [17–19]. In this case, one expresses the generators of a quantum algebra with the generators of the q-deformed Bose algebra and makes sure that all the commutation relations remain. In other words, the quantum algebra is regarded as a subalgebra of the q-deformed Bose algebra. So its representations are naturally subduced by the representations of the Bose algebra. In the above discussion, if one uses the operator algebra on Bargman space in place of the q-deformed Bose algebra, one will get the differential realization of a quantum algebra.

Now, the question is whether one can find another algebra to replace those mentioned above. Recent work on cyclic representations of $sl_q(n+1)$ by Date *et al* [14] implies a positive answer to this question. In their paper, the associative algebra Z^n over $\mathbb{C}$, which is generated by X, Z and I satisfying

$$ZX = qXZ \qquad Z^N = X^N = 1 \tag{1}$$

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§ Mailing address.

is used. This algebra is closely related to Z^n-model, so we will call it the Z^n-algebra in the following discussion, and denote it by Z^n.

In this paper, we will use the Z^n-algebra to realize the quantum superalgebra $U_q(\mathrm{osp}(2,1))$ generated by V_+, V_- and $q^{\pm H}$ satisfying the commutation relations [20–22]

$$V_+V_- + V_-V_+ = -\frac{1}{4}\frac{q^{2H}-q^{-2H}}{q-q^{-1}} \qquad [H, V_\pm] = \pm\tfrac{1}{2}V_\pm. \tag{2}$$

For convenience, we define

$$e_\pm = 2V_\pm \qquad h = 2H \qquad k^{\pm 1} = q^{\pm h}.$$

Then the algebraic relations among the generators of $U_q(\mathrm{osp}(2,1))$ can be written as

$$\begin{aligned} &e_+e_- + e_-e_+ = -\frac{k-k^{-1}}{q-q^{-1}} \\ &ke_\pm k^{-1} = q^{\pm 1}e_\pm. \end{aligned} \tag{3}$$

Consider a mapping Π from $U_q(\mathrm{osp}(2,1))$ to Z^n:

$$\begin{aligned} &\Pi(e_\pm) = \sum_{m,n=0}^{N-1} A^\pm_{mn} Z^m X^n \equiv \hat{e}_\pm \\ &\Pi(k) = CZ \equiv \hat{k} \\ &\Pi(k^{-1}) = C^{-1}Z^{N-1} \equiv \hat{k}^{-1} \end{aligned} \tag{4}$$

where $\{Z^mX^n = f(m,n) \mid e \leqslant m, n \leqslant N-1\}$ is the basis of Z^n and $A^\pm_{m,n}$; C belongs to the complex field $\mathbb{C}$. Using (3), one can easily prove that if $A^\pm_{m,n}$ and C satisfy

$$\begin{aligned} &A^+_{mn} = 0 \;\; n \neq 1 \qquad A^-_{mn} = 0 \;\; n \neq N-1 \\ &\sum_{m+m'=n \text{ Mod } N} A^+_{m1}A^-_{m'N-1}(q^{-m'}+q^m) = 0 \qquad n \neq 1, N-1 \\ &\sum_{m+m'=1 \text{ Mod } N} A^+_{m1}A^-_{M'N-1}(q^{-m'}+q^m) = -C/(q-q^{-1}) \\ &\sum_{m+m'=N-1 \text{ Mod } N} A^+_{m1}A^-_{m'N-1}(q^{-m'}+q^m) = 1/C(q-q^{-1}) \end{aligned} \tag{5}$$

then π is a homomorphism mapping. We call the image of this homomorphism mapping a Z^n-realization of $U_q(\mathrm{osp}(2,1))$. It is worth pointing out that (1) implies the condition $q^N = 1$. Accordingly, only in the non-generic case can one use the Z^n-realization.

We notice that (5) contains N equations, but there are altogether $2N$ coefficients to be determined. So we can expect there to exist non-trivial solutions. As an example, we write down a simple realization

$$\begin{aligned} &\hat{e}_+ = A^+_{01}X \\ &\hat{e}_- = (A^-_{1N-1}Z + A^-_{N-1\,N-1}Z^{N-1})X^{N-1} \\ &A^-_{1N-1} = -\frac{C}{A^+_{01}}\frac{1}{q-q^{-1}}\frac{1}{1+q^{-1}} \qquad A^-_{N-1\,N-1} = \frac{1}{CA^+_{01}}\frac{1}{q-q^{-1}}\frac{1}{1+q} \\ &\hat{k} = CZ \qquad \hat{k}^{-1} = C^{-1}Z^{N-1}. \end{aligned} \tag{6}$$

In order to obtain the representations of $U_q(\mathrm{osp}(2,1))$ we now turn to consider the representations of Z^n. Obviously, its regular representation $\rho = Z^n \to \mathrm{End}(Z^n)$ can be written as

$$\begin{aligned} &Zf(m,n)=f(m+1,n) && Zf(N-1,n)=f(0,n) && m=0,1,\ldots N-2 \\ &Xf(m,n)=q^{-m}f(m,n+1) && Xf(m,N-1)=q^{-m}f(m,0) && n=0,1,\ldots N-2. \end{aligned} \tag{7}$$

This is an N^2-dimensional representation. Denote the left ideal generated by $(Z-1)$ by L, then on the quotient space Z^n/L, (7) induces a representation

$$\begin{aligned} &Z\bar{f}(n)=q^n\bar{f}(n) && X\bar{f}(N-1)=\bar{f}(0) \\ &X\bar{f}(n)=\bar{f}(n+1) && n=0,1,2,\ldots N-2 \end{aligned} \tag{8}$$

where $\bar{f}(n)=f(0,n)$ Mod L.

This is none other than the representation frequently used in physics. Using the realization (6), from (8) we get an N-dimensional representation of $U_q(\mathrm{osp}(2,1))$:

$$\begin{aligned} &\hat{e}_+\bar{f}(n)=A^+_{01}\bar{f}(n+1) \\ &\hat{e}_-\bar{f}(n)=(A^-_{1N-1}\cdot q^{n-1}+A^-_{N-1\,N-1}\cdot q^{1-n})\bar{f}(n-1) \\ &\hat{k}\bar{f}(n)=Cq^n\bar{f}(n) \\ &\hat{k}^{-1}\bar{f}(n)=C^{-1}q^{-n}\bar{f}(n). \end{aligned}$$

One can easily check that $\hat{e}_+^N=(A^+_{01})^N$ and $\hat{e}_-^N=\Pi_{i=0}^{N-1}(A^-_{1N-1}q^i+A^-_{N-1\,N-1}q^{-i})$, i.e. this representation is cyclic. It is due to the condition $X^N=Z^N=1$.

Before concluding this short paper, we would like to point out that using the cyclic representations given above, one may obtain new R-matrices associated with $U_q(\mathrm{osp}(2,1))$ by means of the method presented in [4].

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